SPACE-TIME REPRESENTATION IN THE BRAIN.
THE CEREBELLUM AS A PREDICTIVE SPACE-TIME METRIC TENSOR

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The concept of space-time representation in the brain is redefined using tensor network theory. We make the following suggestions. (a) In order to deal with the external world, the brain embeds the external space-time continuum into a high dimensional internal space. External space-time events are represented within the CNS in overcomplete, inherently oblique, reference frames where space and time information is detected as a continuum over each coordinate axis. (b) The central nervous system may be seen as imposing a geometry on this internal hyperspace in such manner that neuronal networks transform inputs in a metric tensor-like manner. (c) In order to coordinate movements the cerebellum acts as a predictive motor space-time metric which allows the establishment of coincidences of goal-directed movements of limbs in space-time with external targets.

Space-time dogma and redefinition

Understanding of brain function is often limited by the tacit acceptance of concepts known to be basically inappropriate. An example is the tenet that the brain, in its internal workings, utilizes space-time reference frames similar to those used in classical mechanics.

The fundamental problem may be briefly stated. The process of locating and intercepting moving objects by an animal (a coincidence of the interceptor and a target) is generally described as a coordinated sensorimotor act. While both sensory and motor functions relate to events taking place in external space-time, a concise description of the manner in which space and time information are inter-related in the CNS has not been clearly formulated. One of the reasons for this deficiency appears to be the usage of separate frames of reference for 'space' and 'time'. Such frames are used, for instance, in the ‘Newtonian’ space-time representation, where x, y, z coordinates of a point represent the location in the Euclidean space, while a separate t, established by a clock, serves as a coordinate of the time-point of the event. The Newtonian representation is quite satisfactory in classical mechanics. For instance, if the space-time coordinates of an ordinary moving object are measured, light can be used as a means to establish simultaneity. Indeed, when filming a moving object, the consecutive numbers, labeling the individual frames, can provide a common time measure t for all x, y, z spatial coordinates. Alternatively, in flash photography the light ‘freezes’ time such that simultaneous space coordinates (all belonging to the same t) can be established for all points represented in the photograph.

However, the utilization of separate systems of coordinates for space and time hinges upon the concept of simultaneity. Since within the brain there is no ‘instantaneous’ simultaneity agent comparable to the light, the classical usage of separate space and time coordinates is inapplicable in the case of describing the inner workings of the CNS.

We indicated an alternative approach in two preliminary communications.22,23 In this paper, we elaborate the proposed concept of space-time representation in the CNS, based on Tensor Network Theory.10,20,21 Our central assumption is that the brain works in a tensorial manner: that is (a) the function of the CNS (the activity of neurons) is expressed vectorially, using various frames of reference; (b) a given physical entity (e.g. movement) may be represented by different vectorial expressions (e.g. covariants and contravariants); (c) when such vectors are assigned to the same physical entity, by definition they are related to one another tensorially. Because external entities, such as a coincidence, are independent of the coordinate system in which they are described, we will refer to them as invariants.

Space-timing and cerebellar theories

The necessity of formulating a concept of not just 'timing' but 'space-timing' in the CNS becomes evident when considering cerebellar function. Indeed, space-timing is particularly important with respect to the coordination of rapid and precise muscular movements such as a saccadic eye displacement.27 The mode of command by which space and time information is handled by the CNS is essential to the understanding how, e.g. saccadic eye movements are mation is handled by the CNS is essential to understanding how, e.g. saccadic eye movements are made.

Hitherto, our understanding of the space-time representation in the brain has been conceptually based
on the fundamentally Newtonian 'bang-bang' hypothesis by Braitenberg.\(^1\)\(^2\)\(^3\)\(^4\) The space-timing was pictured as shooting one cannon to initiate a motion and then stopping it by firing, at a precisely timed moment, an opposing cannon.\(^6\) Braitenberg called this scheme a "chronometric control of movement" and the timepiece necessary for this was placed in the cerebellum which was to be a "clock in the millisecond range".\(^2\) Braitenberg's concept is regarded as a brilliant pioneering effort\(^4\) that evolved through almost a decade.\(^2\)\(^3\) It started with a space-time concept not inconsistent with our approach.\(^3\) However, his hypothesis did later adopt the classical mechanical point of view, since in its final form it related the cellular morphology of the cerebellar cortex to an assumed 'clock-function'.\(^2\) It is noteworthy that up to now his scheme has been the only cerebellar model to consider the problem of space-timing, although this is a key element of the function of cerebellum: motor co-ordination.

The assertion of a clock-function received a measure of support experimentally by the finding that following peripheral stimulation, due to the conduction time along the parallel fibers, Purkinje cells in the cerebellar cortex demonstrated precise time delays as they were sequentially activated.\(^5\) Nevertheless, the clock-work view of cerebellar function, has difficulties even if it were conceptually sound. This is so because (i) mossy fiber afferent inputs are not synchronous; (ii) the fibers have different conduction velocities, signals being spread over 10–20 ms;\(^16\) (iii) a given volley can activate different sets of parallel fibers all reaching the same Purkinje cell with quite different latencies. All the above would contribute to a significant blurring in the time coordinate axis. However, while these facts seriously complicate models based on Braitenberg's hypothesis,\(^9\) the most profound limitation of his approach is the inapplicability of a separable space and time frame.

The fundamental problem is that the signal propagation speed along the axons of nerve cells is not several orders of magnitude faster than the speed of the 'timed' movement, but in fact, has the very same order of magnitude. This is in contrast to cases in which the concept of simultaneity can be applied, where there is a practically instantaneous synchronizing signal (that is fast enough to arrive at the clock from all 'simultaneous' event points within the resolution of the central clock). The case of the CNS is comparable to taking a picture of a moving object, not with an instantaneous flash, but replacing the light with a set of axons (each having a different conduction time). Creating an internal 'picture' of the external reality in the CNS in such a manner, through differently delayed neuronal signals, simultaneous external events will not be represented in the CNS as simultaneous. Conversely, simultaneous onset of firings of a group of neurons with different conduction times will not produce a set of simultaneous external events either. 'Simultaneous occurrence' can be detected by a clock only in the case where an instantaneous (or otherwise synchronous) access is available to it from the 'timed' events. Since in the case of CNS the difference between the speed of controlled events (e.g., movements) and that of the controller signals (slowly propagating neuronal firings) is not great enough to allow for instantaneous and synchronous access to a clock through the axons, simultaneity cannot be established; therefore this concept is not applicable to the internal functioning of neuronal systems. To give a vivid example for the problem of space-timing by a central clock, consider an attempt of coordinating the position of speeding battle tanks from the headquarters, not with instantaneous radio signals but by cavalry messengers.

It is clear that if no superfast command signal is available, then an alternative mode of space-timing must be found that does not rely on the concept of simultaneity. In the same vein, since within the CNS simultaneity of external events could not be established by the use of any 'brain clock' even if such a device were to exist, therefore, the brain must be using an alternative mode of space-timing.

**Conceptual alternatives of space-time representation**

The establishing of space-time coincidences is a basic biological skill; the survival of animals often depends on their ability to generate rapid, goal-directed movements such as in feeding or escaping. The capture of a target by an interceptor, however, can also be described in a more abstract manner that leads directly to a new conceptual model of unified space-time representation in the brain.

A capture is a coincidence of events in which both the target and interceptor merge into a single event-point. This is an invariant, known in physical sciences as a four-dimensional Minkowski-point or a world-point.\(^14\)\(^26\)\(^28\)\(^31\)\(^37\) The understanding of space-time events as invariants is fundamental in both physical and biological sciences, since interactions usually require that events coincide.

From this point of view, the question of space-time coincidences is reduced to the problem of the representation of an event-point (in the external world or in the CNS). This is a fundamental question of abstract geometry; how can coordinates be assigned to an entity which is, by its nature, invariant to coordinate systems.

The problem is well expressed from contemporary physics, which holds the separation of space and time to be arbitrary. According to this view, the four space-time coordinates of an event need not be separated into two groups, one for 'space' and another for 'time': "... space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."\(^11\)\(^4\) In the rest of this paper, we elaborate the view that although neuroscience has hitherto adhered to a classical mechanical point of view it really ought to adopt the modern physical concept.
FUNDAMENTALS OF THE TENSORIAL APPROACH TO SPACE-TIME REPRESENTATION IN THE BRAIN

Basic notions on the tensorial representation of invariant events

Our approach is based on the consideration that a space-time coincidence is a phenomenon, inherently independent of whatever system of coordinates may be arbitrarily assigned to it, that we call an invariant. We further assume that in the CNS an activity pattern of \( f_1, f_2, \ldots, f_n \) firing frequencies over a set of \( n \) neurons can represent, internally, the same invariant that is represented externally by Cartesian space and separate time coordinates.

The distinction between a Newtonian approach and the one introduced here becomes evident as follows. An event-point is usually described in a Newtonian frame of reference by a mathematical vector (an ordered set of real numbers) \( M(x, y, z, t) \). In the CNS the same invariant may be represented by another mathematical vector \( F(f_1, f_2, \ldots, f_n) \), which is a different ordered set of quantities. The concept put forth here relies on the fact that the two vectorial descriptions (\( M \) and \( F \)) are equally appropriate, i.e. neither is, a priori, pre-eminent. Given that both mathematical vectors are assigned to the same invariant, it follows that \( M \) and \( F \) are tensorially related to one another. This consideration provides a basis for a tensorial treatment of the space-time representation in the brain.

From a mathematical standpoint the tensorial approach views the brain in terms of abstract geometry. The intrinsic functional geometry of the CNS hyperspace (the multidimensional space over the points \( F \)) is an internal representation of the external physical geometry (the latter existing over the set of points \( M \)). Thus, the adequate mathematical approach is that of related geometries: one in the four-dimensional physical space externally (usually represented by Euclidean geometry), and another, structural geometry within the CNS networks. Still another is the functional geometry in the CNS hyperspace, a largely unknown, but certainly not Euclidean, geometry.

The technical steps of geometrically featuring 'brain vectors' and some experimental approaches to reveal the coordinate systems inherent in brain function are described elsewhere.

A concise example of the application of tensor network theory to CNS: the cerebellum as a space-metric

The CNS can be viewed, more geometrico, in several ways and at many levels of abstraction in each. Figures 1 and 2 provide contrasting views of representing the geometries involved in structuro-functional properties of the CNS. The conventional approach is directed at the structural geometry of the brain (Fig. 1). The descriptions in Figs 1A, B, C, aim at establishing the system of spatial relations among the physical components, i.e. of the different types of nerve cells in the network. The alternative tensorial view of representing not only the structural, but also the functional geometry of neuronal networks is given in Fig. 2. The tensorial approach also relates the structural and functional geometries to one another, since it treats them by a method that is capable of unifying structuro-functional descriptions. Thus, Fig. 2 serves as a comprehensive introduction of the application of tensor concept to brain function in this paper.

A most conventional way of representing a neuronal assembly is shown by Fig. 1A. Such a descriptive approach is intuitively clear but has serious limita-
TENSOR NETWORK THEORY: LEVELS OF ABSTRACTION DIRECTED TOWARDS FUNCTIONAL GEOMETRY

A NETWORK

\[ \begin{array}{c}
& v_1 \rightarrow v_2 \\
\text{CN} & \rightarrow & \text{CN} \\
& v_3 \\
\end{array} \]

or

\[ \begin{array}{c}
& v_1 \rightarrow v_2 \\
\text{CN} & \rightarrow & \text{CN} \\
& v_3 \\
\end{array} \]

B ARITHMETIC

\[ \begin{bmatrix}
v_1 & v_2 \\
2 & 4 \\
3 & 2 \\
\end{bmatrix} \rightarrow 
\begin{bmatrix}
v_1 \\
2 \\
3 \\
\end{bmatrix} \]

C ALGEBRA

\[ \begin{bmatrix}
v_1 \\
2 \\
3 \\
\end{bmatrix} \rightarrow 
\begin{bmatrix}
v_1 \\
2 \\
\end{bmatrix} \]

D VISUALISATION

\[ \begin{array}{c}
\text{invariant } P \\
\text{covariant } v_1 \\
\end{array} \]

E TENSOR ANALYSIS

\[ \text{Metric: } g^{ij} : \begin{bmatrix} v_1 & v_2 \end{bmatrix} \otimes \begin{bmatrix} v_1 & v_2 \end{bmatrix}, \text{ cofactor } g_{ij} = \text{determinant } g_{ij} = \sum_{k=1}^{n} \frac{\partial v_k}{\partial x_l} \frac{\partial v_k}{\partial x_j} \]

F LANGUAGE

Cerebellar Network, Acting as a Metric Tensor, Transforms Covariant Intention into Contravariant Execution

Fig. 2. Levels of abstraction in the representation of the functional geometry of neuronal networks by tensor network theory. A concise demonstration of how a set of neuronal networks (A) can perform a transformation to be described by a general tensor, such as the metric (E, F). The 2-dimensional euclidean visualization of the vector transformation (D) is a didactic convenience, utilizing the heuristic power of graphics. A: Two of the infinite number of possible network implementations of a quantitatively identical transformation from a multiple input to a multiple output. CN: cerebellar nuclei. B: A numerical matrix expression of the transformation via the networks shown in A. A scaling factor of 1/3 is included to make possible a common quantitative handling from A to D. C: Expression of the transformation-matrix for a whole class of two-dimensional rectilinear oblique reference-frames where the axes are at a variable angle \( \alpha \) to each other. D: visualization of the functional purpose of the particular transformation implemented by the networks. The transformation implements a covariant-contravariant conversion of one kind of vectorial expression into another, both vectors being assigned to the same invariant physical entity \( P \). E: reference-frame invariant expression of contravariant metric tensor: a general formula for covariant-contravariant transformation of vector components for all and any systems of coordinates. F: verbal expression of the tensorial interpretation of cerebellar function.

In contrast to the conventional approaches featured in Fig. 1, Tensor Network Theory aims both at describing the structural geometry and at understanding the functional geometry of neuronal networks. Furthermore, these goals can be achieved at several levels of abstraction (Fig. 2). The abstraction has to commence with a quantitative representation of the network. Thus, starting from the stage shown in Fig. 1C, the reflex arc logogram is made first into a network where the transformations from a multiple input to a multiple output are quantitatively traced. A pair of such parallel networks is shown, in a schematic manner, in Fig. 2A. Both circuits implement a quantitatively (and, as we point out later, qualitatively) identical transformation of a \( r_i \) input vector into an output vector. Note that exactly the same transformation can be implemented not only by the two networks depicted in Fig. 2A, but by an infinite number of other particular network variations. With this approach, the group of all such networks capable
of performing the same numerical transformation can be summarized by a matrix as illustrated in Fig. 2B. Such an array of quantities is an arithmetical abstraction of the function of particular networks shown in Fig. 2A. Expressed only by an arithmetic formula of a matrix, the function of a network appears to be merely to transform an input vector into an output vector, where both vectors are given in some (unknown) particular frame of reference.

If one knew the coordinate axes, such vector-transformations could be expressed at a higher level of abstraction in a more general form which applies to a set of frames of reference where the angle $A$ between the axes is variable. Such an algebraic expression of the transformation-matrix, as a function of $A$, is shown in Fig. 2C. Because there is no reason to assume that the CNS is limited to orthogonal coordinate systems, general (non-orthogonal, i.e. oblique) frames of reference are used throughout Tensor Network Theory.

The algebraic expression in Fig. 2C defines the transformation-matrix implemented by the networks shown in Figs 2A, B in a relatively general manner. First, by selecting a particular value for the variable $A$, a concrete numerical matrix is specified. Then, there are an infinite number of network implementations of a simple particular matrix. Still, neither the arithmetical, nor the algebraical expression of a matrix reveals the functional advantage gained from the vectorial transformation implemented by the matrix. In short, the numbers in the matrices do not explain why a transformation is necessary.

The functional meaning of the transformation through all networks in Figs 2A, B, C becomes evident by a visualization of the vector transformation (Fig. 2D). It is intuitively clear there, that the input and output of the matrix transformation are two different kinds of vectorial expressions both assigned to one and the same physical location $P$, an invariant. The components $v_i$ of the input vector are covariant (they are obtained by the orthogonal projection method) while the components $v^j$ of the output vector are contravariant (obtained by the parallelogram method). The differences between co- and contravariant vectors (and their tensorial relation through the metric) are of cardinal importance in the Tensor Network Theory of CNS. Since the present work is built upon this conceptual foundation, it must be clearly understood before proceeding (cf. ref.21). The significance of defining the covariant or contravariant nature of a vector when using a non-orthogonal coordinate system is illustrated in Fig. 2D. Note that not only the numerical values of the components are different in each version of the vector, but they also have visibly different features. For instance, the covariant components can be established independently of one another, but they do not add up to the invariant. These features are opposite to those of the contravariants. These profound differences provided the basis for us to postulate21,22 that sensory systems in the CNS are using expressions of covariant type while motor systems use contravariant-type components. These two vectorial versions warrant an important distinction. Indeed, because the CNS need not be limited to the use of orthogonal systems, it is meaningless to introduce vectorial notation into neuroscience without specifically characterizing the components of each vector as covariant or contravariant, or else proving that the implied frame of reference is indeed orthogonal. It is apparent that the transformation from $v_i$ to $v^j$ in Fig. 2D expresses a general relationship that exists not only for the specifically depicted $v_i$ and $v^j$, but for all co- and contravariant pairs of vectors. Such a relation exists in every frame of reference, regardless of the directions or number of axes in the coordinate system.

The covariant-contravariant vectorial relationship may be expressed in a totally coordinate-system-free general manner by the mathematical device $g^{ij}$, the so-called metric tensor. The metric tensor transforms a covariant vector into its contravariant counterpart. This operation does not only occur in the two-dimensional space illustrated in Fig. 2D, but may occur in any $n$-dimensional hyperspace. In fact, we use two-dimensional Euclidean geometry here only to facilitate the visualization of the vectorial relations that can exist in multidimensional, non-Euclidean CNS hyperspaces, where not rectilinear (or even not linear) non-orthogonal reference frames may be used.

The contravariant metric tensor is formally expressed in tensor analysis notation in Fig. 2E. This expression, while not intuitively obvious, is immensely powerful; it is devoid of the limitations of using particular reference frames. The coordinate-system-free tensor notation $g^{ij}$ encompasses, in a generalized form, all particular expressions given in any frame of reference. Thus, tensor analysis appears to be the most appropriate language to describe CNS functions, since it can deal, in a general quantitative manner, with such functional properties of networks that are not evident in particular expressions.

Finally, Fig. 2F is a verbal description of the tensorial function implemented by a group of neuronal networks. The advantage of such concise definition as, for instance, “the cerebellum acts as a metric tensor of the motor hyperspace”, is that it expresses in words a precise mathematical statement.

An additional advantage, provided by the use of tensor analysis, is that these abstract expressions can be made concrete at any of the various levels of abstraction, including the ‘concrete’ implementation by biological or man-made neuronal networks (cf. also Fig. 9).

Expansion of the concept of metric: tensorial interpretation of a sensorimotor system model

For the remainder of this paper, two-dimensional Euclidean illustrations (as in Fig. 2D) are used to broaden the application of the tensor concept to encompass several aspects of the sensorimotor system.
Fig. 3. A demonstration, in a symbolic sensorimotor system, of the differences and the relation of sensory and motor coordinates. A: As the simplest general model of sensorimotor systems, an identical pair of two-dimensional sensory and motor reference frames are presented with general (oblique) coordinate axes at a 120° angle. The sensory coordinate system is composed of two sensors yielding independently-established covariant components $v_i$, the vector being assigned to the reference-frame invariant physical entity of a location $P$. The motor coordinates $v^j$ are expressed contravariantly in the same frame of reference (since they physically must add up to the invariant). The $g^{ij}$ contravariant metric tensor expresses the transformation from the sensory covariant components into motor contravariants. Sensory- and motor-frames of reference are rarely identical in biological systems. The model of acoustic sensory and limb motor frame (in A) is to stress that the same version of the vectorial expressions cannot be used in both sensory and motor systems even if the frames of reference were identical (cf. B). B: Dysmetric transfer. Covariant sensory components used directly for motor execution. Without a metric transformation the covariants yield a distorted motor performance. C: Covariant-contravariant transformation through a matrix expressing the metric tensor, resulting in a proper motor execution. The network, implementing the metric tensor $g^{ij}$ is shown by a symbolic neuronal circuitry, comprised of two parallel fibers connecting to each of the two Purkinje cells, which, in turn, send their axons to both cerebellar nuclear cells shown by open and full circles (for other possible network implementations of the same matrix see Fig. 2A). Neither the sensory nor the motor signals in this figure are permitted to incorporate any time delay: all signals here are synchronous, referring to simultaneous events. The activities of the neurons in the network are shown as a function of time in the bottom part of C.
As suggested earlier,\textsuperscript{21,22} sensorimotor coordination can be conceptualized as a transformation from a sensory to a motor vector within general (non-orthogonal) systems of coordinates (see Fig. 3). Components of sensory information are established independently of one another; that is, if sensory information is expressed vectorially, it is of the covariant type. Motor execution, in contrast, is expressed as the resultant of physical components, i.e. a vectorial expression of motor execution is of the contravariant type. Therefore, a transformation from sensory- to motor-information is a covariant-contravariant conversion. Thus, any sensorimotor system capable of expressing an invariant, both from a sensory and a motor aspect, must contain implicitly or explicitly at least one contravariant metric which transforms one type of vector into the other.

A simple model of a sensorimotor system containing such a metric tensor is provided in Fig. 3A. Suppose that the goal of this system is to locate and intercept a moving target, which at time \( t \) is at the physical location \( P \). If the target were a sound source, a 'sensory' system could be implemented by two linear arrays of microphones along the directions of the axes, measuring the orthogonal projections; i.e. covariant \( v_1 \) and \( v_2 \) components of the OP distance from the origin to \( P \). A 'motor' system could be a mechanical device consisting of two rods that can be advanced to any length determined by the \( u_1 \) and \( u_2 \) contravariant motor coordinates. Assume, as the simplest case, that the sensory and the motor frames of reference are identical, i.e. both constitute the same angle \( A = 120^\circ \) throughout this paper. A simulation of a biological equivalent of such a simple system is shown in Fig. 3A, featuring two identical frames of reference, a sensory (auditory) and a motor (limb). It is obvious that biological systems utilize reference frames different from that shown in the model (e.g. biology is vastly more complicated). The point of this rudimentary model is to serve as a demonstration of an extremely important fact. Even if the sensory and motor systems have only two dimensions, and identical frame of reference, the sensory information cannot be directly applied to execute motor actions. If the sensory components that locate the source were transmitted unchanged to the motor system, 'dysmetric' or 'ataxic' performance would result (Fig. 3B). Indeed, while the movement of the target \( P \) is circular, the executed movement of \( P' \) is an ellipse. Thus, the sensory \( v_i \) components cannot be equated with the \( u_i \) motor components; a covariant- to contravariant metric is required for this transformation to be correct.

Later, we will show that following our scheme most sensorimotor systems may contain at least two additional complicating factors which refer to the differences between the sensory and motor coordinate frames: (i) their arrangement (the directions of the axes in the sensory and motor coordinate systems may not be identical) and, (ii) their dimensionality (the number of coordinate axes in the sensory and motor reference frames may be different).

This is the case, for example, in the gaze-stabilizing system of CNS; when the body moves, the eyes and the head move in a compensatory manner. The sensory and motor frames in this system are different (even if only the vestibular and oculomotor frames are considered). In the fully general case of gaze-stabilization, the motor apparatus includes the neck muscles as well as the eye muscles; thus, in such a system the sensory and motor frames are vastly different both in the number and the direction of the coordinate-axes. The symbolic scheme in this paper is intended as a model and a suggestion how to treat tensorially any such sensorimotor system that employs different input-output frames (such as vestibulo-ocular reflex, gaze-stabilizing system, etc.).

The transformation matrix needed to convert covariant sensory information into contravariant motor signals can be implemented by a 'neuronal network' in Fig. 3C. With this network serving as a space-metric, the circling movement of the target is not only properly detected, but an identical movement is faithfully executed. However, the transformations shown throughout Fig. 3 are synchronous; i.e. it is assumed that the vector components are free of temporal delays. Since such synchrony is an assumption that is in conflict with the presence of a variety of conduction-times in signal transmission and processing, a problem is raised which will be considered below in regard to additional features of the metric.

**UNIFIED SPACE-TIME REPRESENTATION**

**IN THE CNS**

The coordinates of the space-time continuum in a sensorimotor system with non-uniform conduction times

Conduction time differences are usually disregarded when global brain function is modeled. That is, one assumes that the system is synchronous in spite of the fact that different neuronal channels have different conduction velocities and lengths which ultimately result in different conduction times. While in experimental analyses of particular subsystems of the CNS delays are usually taken into account, they are interpreted as temporal phase-differences of the neuronal responses, usually to sinusoidal test-functions. Such an approach leaves the development of a conceptually coherent interpretation of space-time representation not only lacking, but almost hopeless (cf. Fig. 4).

In the terms introduced in the scheme of Fig. 3, the delays mean that each of the \( v_i \) coordinate components represents not only the space position of the target, but also the moment in time when each particular position occurred. Thus, the function of the space-metric shown in Fig. 3 has to be expanded to account for time delays; i.e. the network must function as a space-time metric tensor. This is shown in Fig. 4.

In this figure the delayed covariant components of
METRIC TRANSFORMATION OF ASYNCHRONOUS (DELAYED) SENSORY COVARIANTS

A SPACE METRIC ONLY

temporal delays transformed into spatial distortion

B WITH TEMPORAL LOOKAHEAD: SPACE-TIME METRIC

covariant-contravariant transformation with asynchronism

Fig. 4. Transformation, through a space-metric, of asynchronous space-coordinates, contrasted with the transformation via a space-time metric. A: Distorted performance through space-metric only. The sensory covariant components \( t \) incorporate individually different delays \( d_i \) in detecting the location of the circling target. If these covariants are taken as referring to simultaneous events and are transformed by a network that represents the metric tensor of the space only, then the motor execution is distorted into an ellipse (right side). B: Proper performance through a predictive, space-time metric tensor. The delayed covariants undergo a temporal 'lookahead' procedure thus yielding a set of temporally extrapolated signals, (shown by dotted lines in the bottom of B). For the 'temporal lookahead' procedure the 0th, 1st and 2nd time derivatives of the covariant sensory signals have to be taken by a 'stack' of neurons shown only symbolically in the middle section of B. From such a 'lookahead-module' the covariant sensory signal is temporally extrapolated on neurons of the cerebellar nucleus that take the weighted sum of derivatives. The activities of the neuronal elements are plotted in the bottom part of B by continuous lines.
\( \nu_t \) do not simply encode the position of the target at time \( t \), but rather, each \( \nu_t \) component represents the position of the target at the time-point of \( t - d_i \), where each delay \( d_i \) may be different.

From an experimental point of view, the schemes in Figs 3 and 4 also include a temporal display of the activities of the neurons in the transformation-network (bottom of both figures). Note, that all neurons are activated sinusoidally, only their amplitude and phase are different. This scheme illustrates how formidable is the problem of inferring, from such single cell data, the global functional properties which emerge from the compound activity of single elements. In Fig. 4A (bottom), the covariants and contravariants are shown together both for the case of simultaneity (comparable to Fig. 3C: the signals shown by dotted lines) and for the case of delayed, non-simultaneous signals (shown by continuous lines). Again, the difference between the simultaneous and non-simultaneous cases would only be detectable experimentally in the phase of the sinusoidal time functions.

In contrast to displaying single cell activities, a visualization of the 'assembled' vectorial output itself (right side of Fig. 4A) makes it clear that if the metric transformation in such an asynchronous system were performed directly on the delayed components, the contravariants would yield a distorted, elliptical movement instead of the circular one. Here, the existence of some kind of interaction between space and time information in neuronal systems is clearly indicated: temporal dispersion may be transformed into spatial distortions. Such a functional interdependence of space and time is well known in psychophysics. For example, the Pulfrich illusion (cf. ref. 36) demonstrates that temporal differences in the detection, by the right and left eye, of a planar pendular movement make the movement appear to follow a spatially distorted elliptical path. On a motor level, there is a similar relationship between the temporal spike responses of cerebellar neurons to spatial precision of saccadic eye movements. The existence of some kind of interaction between space and time is well known in psychophysics. For example, the Pulfrich illusion (cf. ref. 36) demonstrates that temporal differences in the detection, by the right and left eye, of a planar pendular movement make the movement appear to follow a spatially distorted elliptical path. On a motor level, there is a similar relationship between the temporal spike responses of cerebellar neurons to spatial precision of saccadic eye movements. In the field of psychophysics the mismatch of the subjective perception of objective temporal sequences is generally known. A manner in which the CNS may establish a relation between space-time coordinates is proposed below.

**Predictive space-time metric: reestablishing the relation by temporal lookahead between an event and its synchronously delayed coordinates**

The present solution to the problem of establishing space-time coincidences via asynchronous CNS function is based on the predictability of the future values of vectorial components (Fig. 4B). Such a ‘temporal lookahead’ scheme was elaborated earlier, based on experimental evidence indicating that the activity of Purkinje cells shows zero-, first-, and even the second-order time derivatives of the velocity-stimulus (cf. Fig. 12 in ref. 11). The basis of the principle of prediction is that a set of derivatives (according to the coefficients in a Taylor-series-like expansion) of the input time-function can be produced by a ‘stack’ of Purkinje cells which are activated by a ‘beam’ of parallel fibers. If the activity of these Purkinje cells is summed on a cerebellar nuclear neuron, then a temporal lookahead of the input signal can be obtained (for a detailed explanation of the functioning of such a ‘lookahead module’ see Fig. 1 in ref. 20). We have suggested there that such ‘prediction’ could be the neural basis for coordinated motor actions when body movements have to anticipate the location of fast moving targets. Recently, features of temporal extrapolation in CNS have been corroborated with experimental evidence.

It is proposed that the metric network utilizes the ‘lookahead modules’ as follows: the sensory part of the scheme in Fig. 4B is initiated by a set of delayed covariant components. Then, if a ‘lookahead-module’ is incorporated by a network (shown symbolically by a few Purkinje cells at the center of Fig. 4B) then both \( v_1 \) and \( v_2 \) can be processed in a manner such that the individual delays \( d_1 \) and \( d_2 \) are compensated by an identical ‘temporal lookahead’ \( d_1 \) and \( d_2 \), respectively. As a result, the synchrony of covariant components may be reestablished in the CNS so that a simple space-metric is then usable.

A space-time metric tensor in the brain, therefore, must serve two simultaneous purposes: (a) to provide a procedure by which each of the temporally-dispersed covariants are augmented by a suitable ‘temporal lookahead’ so that they all refer to the same external time-point, and (b) to assemble the invariant from such temporally-extrapolated covariant components.

The space-time metric in Fig. 4B achieves these goals by (a) containing ‘lookahead-modules’ for each covariant space-time component in order to compensate for the delay involved in the individual coordinate, and (b) providing a connectivity-matrix (just as in Fig. 4A) which serves as a metric.

However, a space-time metric tensor may not be as simple as shown in Fig. 4. Firstly, in some particular expressions of the network, individual neurons in the first set may not be separable from the neurons in the latter. Thus, in general, a space-time metric may be implemented by a morphologically-unified network. Second, since the matrix of the metric tensor in an oblique frame contains non-zero off-diagonals, a network expression of such matrix will have cross-connections from the \( i \)th input to \( j \)th output line (\( i \neq j \)). Therefore, a ‘slicing’ of the network implementing the space-time metric into separate input-output loops is also inapplicable in the general case. But before a more detailed discussion of such general features of morphologically-unified metric networks, we have first to elaborate further the basic scheme of the space-time metric.

*Explicit* and *implicit* space-time metrics

The fundamental scheme in Fig. 4B is expanded
A COORDINATES IN SPACE AND IN TIME

time $\rightarrow$ t - d_l $\rightarrow$ lag $\downarrow$ sync. $\downarrow$ t + d_l $\rightarrow$ lead

t_1

t_2

d_2

B EXPPLICIT (SENSORY AND MOTOR) SPACE-TIME METRIC

\begin{align*}
&v_i(t) \\
&v_i(t-d_1) \\
&v_i(t) \\
&v_j(t) \\
&v_j(t+d_1) \\
&v_j(t)
\end{align*}

sensory delays s. lookahead metric m. lookahead motor delays

---

Fig. 5. Circuitry schematics of an explicit space-time metric tensor, implemented by a neuronal network. The function of temporal lookahead modules is to compensate for delays both in the sensory and in the motor pathways. A: The space and time information inherent in neuronal signals existing at time $t$ is shown in Newtonian space-time frame. The centralized clock-time is shown in the top row of clocks, the Euclidean space-coordinates are shown in the second row of oblique frames. In the vectorial symbolism (third row) the covariant $v_i$ and contravariant $v^i$ components are appended with time-indices, representing the clock-times that the vectorial components refer to. These vectorial components are carried by neuronal signals in the circuitry shown in B at the locations marked by arrows. B: Network implementation of the explicit space-time metric. Each of the 'stacks' of five Purkinje cells symbolize a 'lookahead module'. The sensory covariants first undergo a temporal lookahead $d_1$ and $d_2$ to compensate for the corresponding delays in sensory paths. This is followed by a metric transformation into contravariant components. Lastly, the components of the motor vector are also extrapolated in time, to compensate for the delays $d_1'$ and $d_2'$ occurring in the outgoing motor paths.

---

If paths in the sensory system contain delays $d_i$, the sensory signals arriving at the central processor at time $t$ will not be those components of the covariant vector $v_i(t)$ that belong to the coincidence-time $t$, but the 'lagging' components of $v_{(t-d_i)}$ which refer to the locations of the target at the times $t - d_i$. Thus, the $v_{(t-d_i)}$ components have to be advanced by a 'lookahead' via the Taylor-assemblies to provide an identical set of predictions $d_i$ that cancel out the delays $d_i$. Such a predictive 'lookahead module' is schematically depicted in Fig. 5B and later in the paper by a stack of Purkinje cells connected to a cerebellar nuclear neuron (from the computer model of Fig. 1 in ref. 20). If a simultaneity of the covariant components of $v_i(t)$ is re-established in the CNS (all components referring to the space-time event point at $t$), the covariant-contravariant transformation from $v_i(t)$ to $v^i(t)$ can be implemented by a network-matrix serving as a metric tensor. However, assuming that the contravariant components are carried by motor pathways that imply delays $d_i'$, the space-time coincidence-point at $t$
**IMPLICIT (SENSORIMOTOR) SPACE-TIME METRIC**

\[
V_i(t) \quad V_i(t-d_i) \quad V_j(t+d_j) \quad V_j(t)
\]

\[
\Omega = t - d_i
\]

\[
d_1 + d_1' = t + d_j = T
\]

\[
d_1 + d_2
\]

\[
d_2 + d_2'
\]

\[
g_{ij} = V_i(t) \cdot V_j(t)
\]

**Fig. 6.** Circuitry schematics of a network representation of an implicit space-time metric. The scheme is derived from Fig. 5 by contracting each pair of sensory \(d_i\) and motor \(d_j\) lookahead-modules into a single predictor module (yielding \(n \times n = n^2\) modules altogether, contrasted to the \(2n\) number of modules in the scheme of Fig. 5). Each lookahead-module (e.g. connecting the \(i\)th input to the \(j\)th output) provides a \(d_j = d_i + d_j^i\) temporal extrapolation. As a result, both the input and the output of such a network are asynchronous: the input vectorial components refer to \(t - d_i\) external clock-times (they lag \(t\) by different delays), while the output signals refer to \(t + d_j\) time-points (they lead \(t\) non-uniformly). No vector within such network represents external simultaneity.

A central conclusion of this paper is, however, that there is no theoretical need to separate the input and output delay compensations from the metric transformation itself. Figure 6 shows a compact scheme in which the input and output lookahead modules of Fig. 5 are combined into a unified network implementing an implicit metric where the matrix elements (connecting lines between each input and output) themselves produce the required temporal prediction properties.

The transition from Fig. 5 to Fig. 6 is based on a contraction of the lookahead-modules. The input delay \(d_i\) and the output delay \(d_j\) in the path from the \(i\)th input to the \(j\)th output were combined so that the \(g_{ij}\) matrix-element (the number of connectivity-lines from the \(i\)th input to the \(j\)th output) incorporates a delay \(d^{ij}\), where

\[
d^{ij} = d_i + d_j.
\]

Such 'compaction', while technically simple, introduces a significant conceptual change in the interpretation of the functioning of the network. It is a fact that in target interception the vectorial expressions at the sensory and motor coordinates, \(v_i(t)\) and \(v_j(t)\), respectively, refer directly to an identical space-time event point (cf. Fig. 5A). However, due to the delays
in the sensory and motor pathways, both the $v_{i(t-\delta)}$ input and the $r_{\delta^{i*}}$ output of the central network are asynchronous. Thus, in the implicit metric (in contrast to the explicit system) the components of any given CNS vector need not represent external simultaneity. The 'implicit' sensorimotor system does not separate a sensory and a motor metric, nor does it re-establish the external simultaneity anywhere within the network. Such a network, as in Fig. 6, represents the true unified character of sensorimotor space-time metric in the CNS.

A concise intuitive interpretation of the functioning of such space-time coordinator networks can be provided based on the heuristic power of the metric tensor. Basically, this mathematical device expresses the relation between any invariant entity and its description by coordinates. It gives the algorithm for the internal reconstruction of the invariant entity itself from the distributed vectorial components ascribed to it. For example, given an OP distance (a line-element, which is an invariant of the physical world), the $n$ coordinate components which describe OP are not invariant quantities (they depend on the frame of reference and also on the method by which the either covariant or contravariant components are obtained). While more than one feature (e.g. the $x$ and $y$) of a physical invariant (such as OP) may be implicitly present in a single $r_t$ value, these different features are encoded, ‘scrambled’ together in every $r_t$. Because of this method of ‘scrambling’, no single component may carry explicit information about a separate feature. Furthermore, no single component contains enough information to recreate the invariant. On the other hand, the full set of components, e.g. the $r_t$, co-variants, does contain the total information. However, to obtain the invariant, all the $r_t$ components have to be assembled in a well defined manner. This particular ‘manner’ is given by the metric tensor: its matrix provides the coefficients (weights) of all the pair-products of the components $r_t$ such that the sum of these weighted pair-products yields the physical invariant itself.

The ‘scrambled’ encoding can be illuminated through the notion of orthogonality. Indeed, in connection with our two-dimensional Euclidean illustrations, it is worth considering that the most fundamental (orthogonal) algebraic example of all ‘metrics’ is the Pythagoras theorem. It is basic knowledge that in a Cartesian frame a displacement $D$ is encoded by both $x$ and $y$ in such a way that in order to reconstruct the invariant $D^2$ the components must be put together by adding their squares. This can be expressed not only algebraically but also by vector-matrix symbolism and finally by the reference-frame invariant tensor formalism:

$$D^2 = x^2 + 0.\,xy + 0.\,yx + y^2$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} (x, y) = g^{ij} r_i r_j.$$
CEREBELLAR NEURONAL NETWORK ACTING AS A SPACE-TIME METRIC TENSOR

Fig. 7. Circuitry layout of an implicit space-time metric tensor. The network is identical to the one shown schematically in Fig. 6, only the physical layout of the network is different. Such morphologically realistic representation is necessary in order to demonstrate that the abstract conceptual scheme does, indeed, confer with the morphology of existing neuronal networks in the brain. The input vector to this scheme is $o_{T}$, a covariant two-dimensional vector expressed in sensory frame, the components being asynchronous, referring to $Q = t - d_i$ external time-points. The output is a contravariant vector $dT$: a two-dimensional vector expressed in an identical motor frame, with asynchronous components referring to $r = t + d_j$ external time-points. The network implements a transformation of the input vector to the output, expressing a contravariant space-time metric tensor $g_{ijT}$. 
expressed by $g^{ij}$, that is, the cerebellar space-time tensor changes the lower $i$ space index to an upper $j$ and the lower time index $\Omega$ into an upper $T$ index.

It would be desirable to identify in the CNS those networks that perform 'implicit' unified sensorimotor space-time metric functions (as in Figs 6 and 7), and those that more closely correspond to the expanded, explicit sensory and motor metrics (as in Fig. 5).

Before doing so, it must be pointed out that the reason for the development of an explicit metric in the CNS appears to be economy in the number of neuronal components. Assume, for example, that 100 neurons are required for a single lookahead-module (a reasonable assumption; cf. Fig. 1 in ref. 20). Then, the separable explicit scheme (cf. Fig. 5) would require 'only' $100 \times 2 \times n$ 'differentiator' neurons, since look-ahead-modules are built into the $n$ input lines and into the $n$ output lines. In turn, in the implicit scheme (cf. Fig. 6) the necessary number would be $100 \times n^2$, since all of the $n \times n = n^2$ connecting lines of the matrix contain lookahead-modules. The difference between $2n$ and $n^2$ becomes very significant if $n$ is large. Thus, in contrast to one's intuitive expectation, a conceptually 'compacted' implicit scheme requires an enormously higher number of neuronal elements than an explicit scheme which performs the identical function.

For example, if the space-time metric in question is the cerebellum and the differentiating neurons are the Purkinje cells, then using an implicit network would require one million Purkinje cells in the case of a one hundred dimensional sensorimotor system. By contrast, only twenty thousand Purkinje cells would suffice for a separable, explicit sensorimotor metric. Likewise, for a larger system (e.g. with ten thousand-dimensions) these figures would be ten billion Purkinje cells versus two million.

The one hundred dimensional hyperspace may approximate an amphibian (e.g. frog) cerebellum, while the ten thousand dimensional sensorimotor system may be that of, for example, a cat. The requirement of twenty thousand Purkinje cells for the frog cerebellum is well within the range of morphological reality, whereas one million Purkinje cells is far too large a number. Similarly, a number as large as ten billion Purkinje neurons in the cat cerebellum can be ruled out, while two million Purkinje cells are well within the range of established facts. Thus, according to our view, in very high dimensional full sensorimotor systems, an explicit metric is more likely to be implemented than the implicit variety.

In contrast, for a rudimentary sensorimotor system that acts only on a dozen muscles (e.g. the vestibulo-ocular reflex) even an implicit sensorimotor network is within morphological reality, since $n = 10$ would require only a small nucleus composed of about ten thousand neurons.

**General tensorial architecture of sensorimotor systems**

The possibility that separable 'explicit' sensory and motor metrics are present in sensorimotor systems is a reminder of the fact that sensory and motor coordinates usually differ both in dimensionality and directionality.

A difference in the directionality of the coordinate axes in the sensory and motor systems poses no problem if the number of coordinate axes in the motor system is equal to or less than the dimensionality of the sensory system, i.e. the motor hyperspace is a projection of the sensory hyperspace. In such a case, all that is needed is an additional projection-matrix, $p^4$, which transforms the motor vector expressed in the sensory-type frame (upper index $j$) into the motor frame of reference, where the components have an upper index $k$. Such a $p^4$ can be inserted into the scheme featured in Figs 5-7 by placing it after the metric transformation, so that the overall connectivity matrix becomes $g^{ij}.p^4$, (an $i \times k$ matrix). In this case, the metric transformation and the change of reference-frame can be implemented in one transformation by a single input-output network. The $d^4$ 'motor' lookahead, of course, has to equal the delay in the $k$th output of the motor system. As above, the contraction of such an explicit metric into a single (implicit) metric can be implemented in the same manner as in the case where its sensory and motor frames of reference are identical. In the compacted 'implicit' network a 'lookahead module' has to be incorporated into each of the $i \times k$ connection to yield a temporal prediction of:

$$d^4 = d_i + d^4.$$  

In contrast to the above, the total sensorimotor system cannot be implemented with a unified single matrix (as in Figs 6 and 7) if (a) a specific requirement necessitates an explicit space-time metric tensor and/or (b) there is an increase of the dimensionality from the sensory to motor systems (the motor system has a higher degree of freedom than the sensory). In the latter case, the motor hyperspace is not a projection of the sensory space, but the sensory system (which embeds the physical three-space) itself becomes an embedded space in the motor hyperspace. Separate sensory and motor space-time metric tensors may be a biologically desirable feature in itself for reasons other than the above-mentioned difference in reference frames or because of circuit economy. A fundamental reason for sensor and motor separation is that any CNS subsystem that has to reach realistic conclusions about the external invariant itself must incorporate a separate metric tensor. Indeed, *neither a covariant nor a contravariant vector alone can express such parameters of the invariant as distance*. However, it is known that distance, for example, can easily be expressed by the inner product of the two different types of vectors: $D^2 = v^i.v_j$. Here, the simultaneous availability of both the co- and contravariant vectors implies the availability of the metric tensor, since $v^i.v_j = g^{ij}.v_i.v_j$. Given that the metric tensor summarizes all geometric relations, its presence allows the...
Evidently, some sensorimotor systems work in a manner such that at times only the sensory part is active, producing, on its own, a biologically meaningful decision relating to the invariant. Such is the case, for example, when a visual judgement is made of distance, without the necessity of the motor system (other than the extraocular eye muscles) being simultaneously engaged in this sensory process. The biological purpose of such a sensory system is to incorporate, within itself, an adequate (geometrically homomorphic) internal representation of external invariants.

Using the frog as an example, it is a fact that a jump establishing a space-time coincidence does not follow, automatically, from every visual message; it occurs only when the space-time event (the fly's location) is within range. In our terms, it must be decided that the physical invariant, the distance between the fly and the frog, is within the range of possible jumps. The comparison of such invariants even in this simple case is further complicated by the fact that they are 'distances' of space-time event-points, involving not just space but also temporal aspects such as speed. As a consequence, a jump to a non-moving or an overly fast moving target will not occur even if the target is within spatial range.

The above considerations suggest that sensory perception requires the building of a sensory metric, separable from the motor system, that transforms covariant sensory vectors into contravariant sensory vectors expressed in a sensory reference frame. Indeed, the existence of such a sensory metric was suggested, prior to the tensorial approach, for vision, and color perception. We suggest and will further elaborate below that the sensory covariant-contravariant transformations serve as precursors to the generation of motor intention vectors, also in the sense of permitting fundamental sensory decisions on whether motor action is appropriate.

A general tensorial architecture of a sensorimotor system can be outlined now, based on the above considerations. In a general case the sensory frame of reference is different from the motor frame, the latter being allowed to have any number of dimensions (including a greater number of axes in the motor frame than in the sensory system of coordinates). Another general feature is the possible existence of a separate sensory space-time metric and a separate motor space-time metric. A sensorimotor scheme with the above general features is shown in Fig. 8. Because of the differences in sensory and motor frames, the circuitry schematics require several different vectorial expressions attributed to a single external space-time event-point. In the scheme of Fig. 8, each vector is not only conceptually and geometrically well-defined, but each also corresponds to a biologically well-circumscribed CNS function.

A minimum of four basic vectorial expressions are required since two different frames of reference are used, and in each frame there is a covariant and a contravariant version of the vector. Thus, the principal vectorial expressions of the general functions of a sensorimotor system are:

1. **Reception vector**, defined here as a covariant expression in the sensory frame of reference of the external invariant. In such a vector, the components are derived directly from the invariant and established independently of one another. Such vector, however, contains the information in a 'scrambled' manner, e.g. not only the spatial information is represented along all axes, but these vector components also contain a temporal 'blur'.

2. **Perception vector**, defined here as a contravariant expression in the sensory frame of reference of the external invariant. The existence of both the covariant- and contravariant vectors implies a sensory metric tensor. This contravariant metric provides the basis for building a homomorphic internal model of the external world. If the geometry of the internal representation resembles the external realities, then judgements based upon the internal geometry will provide successful guidance for manoeuvring within the system of relations existing in the external world.

3. **Motor intention vector**, defined here as a covariant expression in the motor frame of reference of the external invariant. This expression is a necessary precursor of motor execution vector, since the components of this vector are the ones that can be obtained from the sensory perception vector by a procedure called covariant embedding (taking the inner products of the perception vector with another vector). The latter we refer to as sensorimotor proprioception vector, defined here as a covariant expression of the unit vectors of the motor frame, in the sensory frame of reference (see inset at lower right of Fig. 8). The procedure of the covariant embedding yields a mathematically unique set of components for the intention vector even if the motor frame has a higher dimensionality than the sensory frame. Obtaining the intention vector by inner product requires only multiplying and summing networks, which are matrix operations that the CNS networks can perform easily (as shown below). The intention vector, although a representation of the external invariant, cannot be directly utilized to generate the motor action because of its covariant character.

4. **Motor execution vector**, defined here as the contravariant expression in the motor frame of reference of the external invariant. The components of this vector, by definition, will be capable of physically assembling the external invariant. However, in order to obtain the contravariant from the covariant intention vector, a motor metric (a contravariant metric) has to be available within the motor system of the CNS.

With the use of these function vectors of the CNS, the scheme of Fig. 8 provides an explanation of the functioning of the sensorimotor system from sensory
Fig. 8. Circuitry schematics of the tensorial architecture of a sensorimotor system where the sensory and motor frames of reference are not identical. This figure serves as a general model for such sensorimotor systems as the vestibulo-ocular reflex, or the more general gaze-stabilization system, involving neck muscles. The four basic vectorial expressions within the CNS (reception, perception, intention, and execution) are all assigned to one and the same external invariant, the location of a moving target. These vectors are represented at four different levels of abstraction. Two-dimensional Euclidean visualization, together with centralized clock-time, is shown in the top row. Tensor notation is used in the second row. Further, a verbal description as well as a network realization of the geometrical transformation is offered. For further explanation, see text.
reception to motor execution, using four different levels of abstraction; the 2D Euclidean illustration, verbal description, tensor notation and network representation. Since this figure is an expansion of the previous figures of this papier, the explanation can rely largely on the earlier schemes, with only few differences. First, Fig. 8 uses different visual symbols for the neuronal elements of the sensory lookahead-modules than the ones used in Figs 5-6. The neuronal elements in Figs 8 and 9 were redrawn from appropriate parts of the original drawings (Figs 102, 275, 323, 330) of Ramón y Cajal. The reason for such representation is that we deem it absolutely necessary to demonstrate that the conclusions of a theoretical abstraction, such as those of the tensor theory, are not inconsistent with the fundamental morphological realities of existing neuronal networks. Just as abstraction in Figs 1 and 2 started with realistic representation of networks, theory has to conclude in yielding exactly such neuronal networks. Another feature in Fig. 8 is that the sensory system is augmented by a simple auxiliary network that multiplies the corresponding covariant and contravariant sensory component-pairs and sums their products, thereby taking the

\[ D^2 = v_i v^*_i \]

inner product. If the threshold of this inhibitory neuron is set at \( D^2 = r^2 \) (\( r \) being the jump limit), then the sensorimotor transduction is blocked whenever the target is not within range.

Strictly speaking, the auxiliary circuit of sensorimotor blocking is not necessary for the functioning of the sensorimotor scheme. However, its inclusion serves two purposes: (a) to demonstrate the simplicity of networks making internal geometrical decisions on external invariants and (b) to raise the issue of inner product implementation, especially since it is needed also for the operation of covariant embedding, the next step of the scheme. Indeed, a sum of products may not be produced by a single neuron in the CNS. A technically significant implementation of an inner product is worth mentioning in this paper. As shown in the covariant embedding part of Fig. 8, neurons with logarithmic input-output characteristics sum up the inputs and the result drives a neuronal output with exponential characteristics. With such an 'embedding module', a local network consisting of only three neurons can circumvent the requirement for multiplicator neurons.

The covariant embedding part of Fig. 8 produces a motor intention vector that goes through a network, representing the \( 3 \times 3 \) matrix of the motor metric tensor, and then through the 'motor lookahead' modules to provide the contravariant execution components. Each lookahead-module is symbolized in Figs 8-9 by only two Purkinje cells.

In short, a network as in Fig. 8 can transform a \( v_{\text{cov}} \) covariant sensory reception vector, expressed in two-dimensional sensory frame and consisting of temporally laquing asynchronous components, into a \( u_{\text{contr}} \) contravariant motor execution vector, expressed in a differently arranged, higher dimensional motor frame, with each component of the asynchronous output having a temporal lead \( d^m \).

The network shown in Fig. 8 satisfies the theoretical requirements from a sensorimotor scheme, and correlates well with several neuroanatomical features of the CNS. The network illustrated in Fig. 8 can also be depicted by a general simplified scheme of the spatial organization of CNS networks as, for instance, in an amphibian (Fig. 9). Comparable classical schemes of the neuronal networks of the regions of the CNS suggest that particular (morphologically distinct) brain regions may be engaged primarily in sensory functions, while others perform mostly motor functions. Indeed, while even the simplest blueprint of the global functioning of the CNS lies further in the future, it seems reasonable to suggest that while a function corresponding to a motor space-time metric is probably implemented by the cerebellum, the superior colliculus may well serve, in lower vertebrates, as a sensory space-time metric.

**DISCUSSION**

Several general comments regarding transformations via tensor networks can be offered. From a technical point of view, a covariant embedding (either of the external points into the sensory space or of the perception-points into the motor space) yields a unique set of covariant components, where the individual components are established independently of one another, regardless of the dimensionality of the embedding system. That is, an embedding space is free to have any number of dimensions; it may be vastly overcomplete compared to the embedded space. In Fig. 8 for example, the physical point of the target-location can be embedded into not just two, but any dimensional sensory space. Likewise, in establishing the covariant intention vector, the motor hyperspace may have any number of coordinate axes, totally independent of the sensory frame. The process of covariant embedding thus provides a solution to the vexing problem of how the CNS arrives at unique solutions even though it has an overly high degree of freedom.

Interestingly, the covariant-contravariant scheme of a covariant embedding followed by a metric transformation also reconceptualizes the old notion of functional reliability of neuronal networks achieved by means of structural redundancy. The orthogonal projection-character of the embedding covariant components (independent of one another), together with the overcompleteness of the embedding hyperspace, yields an increased reliability of the contravariant expression. Firstly, because of the independence of their establishment, errors do not accumulate, the accuracy of measuring one covariant not affecting the accuracy of establishing the other. Secondly, in the case of an overcomplete system, a loss or drastic alteration of some covariant components will not necessarily result in a proportional distortion of the contravariant out-
put, since some of the errors cancel each other out through the interconnections of the metric. The numerical distortion produced by a damage of the covariant input depends on the metric, e.g. on the size of the off-diagonal elements it contains. The further these elements are from zero, the more reliable the system is, but the higher the number of input-output connections which are required for the morphological implementation of the matrix. Thus, while conceptually different, an overcomplete tensorial system resembles the classical scheme of redundant organization in the sense that there is a trade-off between reliability and structural economy in both.

The overcomplete covariant embedding does not only apply to sensorimotor systems. Indeed, as shown in Fig. 9, covariant embedding followed by metric transformation leads to a cyclic scheme of CNS function where the number of dimensions may be arbitrary in each subsystem, and covariant and contravariant vectorial expressions (shown in blue and red, respectively) alternate throughout the scheme.

Methodological implications. This paper conjoins the tenets of our two preceding papers on the tensorial approach to brain function. The first paper introduced tensor network theory and proposed a temporal predictive feature of cerebellar neuronal network.\(^2\) The second established the thesis that the cerebellum acts as a metric tensor transforming the covariant intention vectors into contravariant execution vectors.\(^2\) The present paper provides a synthesis: it proposes that instead of acting only as a space-metric, the cerebellum acts as a metric tensor of the unified space-time manifold working with temporally extrapolated space-time coordinates. This unification is significant also from a technical point of view, as it merges the mathematical devices used in those previous papers. This merge helps the answering of an obvious methodological question regarding the paradoxical adequacy of 'linear' tensor analysis in treating nonlinearity-laden systems.

Indeed, tensor analysis is applied most often to linear problems. However, as is well known, it is quite appropriate in nonlinear theories such as relativity. Equally prominent is the applicability of tensors in the description of nonlinear (only locally linear) tensions in inhomogeneous elastic bodies. In particular, in this paper 'prediction' is suggested to compensate for the temporal delays. Going forward or backward in time are not linear features in the common use of the term. The overall function of the network can still be characterized by a fundamentally linear metric function, because one kind of nonlinearity (look-ahead) compensates for the other (delay). In fact, linearity in any physical system never exists in the true mathematical sense; no physically implemented 'straight line' is linear upon close examination. Thus, in the CNS, just like in any other physically implemented system, the question is not whether linear methods of description are applicable, but to what extent they are usable.

The conceptual unification of space and time handling in CNS suggested by this paper reaches further than the simple merging of methods. Basically, the issue is how to relate our understanding of the geometrical structure of CNS to our understanding of the physical space-time geometry. In mechanics, Newton, by his utilization of distinct space and time frames, established a separation, rather than a merging, of the concepts of space and time. At a philosophical level, the space-time schism (in the sense of attributing separate existence to these two parts of one entity) was completed late in the 18th century by Kant, who held space to be an a priori concept of mind.

The synthesis of the notions of mechanics with those of brain function started relatively recently. While Wiener\(^3\) and McCulloch\(^1\) mentioned that the Newtonian frame is not the only one applicable to living systems, a systematic elaboration of Newtonian space-time representation in the brain had not been offered until Braitenberg's paper.\(^1\) He characterized his approach thus: "the structure of the cerebellar cortex is viewed in the same spirit in which we would analyze an unknown machine".\(^2\) It is true, that the CNS is a 'machine', in the sense that it imposes an order on the moving parts of the musculoskeletal system; it is also evident, that such movements are externally describable by classical Newtonian mechanics. It does not follow, however, that the inner workings of the CNS utilize the same Newtonian mechanics used in the description of the movement. Indeed, since Newtonian mechanics is only one among the several different types of mechanics (e.g. Newtonian, quantum, relativistic) in an unknown machine, both the implementing device (the mechanism) and the set of rules that govern such device (the mechanics) must be taken as unknown.

Fig. 9. Circuitry layout, a tensorial 'blueprint' of the model of an amphibian brain. This network is identical to the one shown schematically in Fig. 8, only the layout of the network is different in order to demonstrate that the tensorial interpretation is fully consistent with the conventional descriptions of the neuroanatomical realities. The correlation of the conventional and the tensorial approaches must be made evident in order to ensure that the descriptive and abstract representations do relate to one and the same reality. Thus, the neuronal elements shown symbolically in this figure have been redrawn from appropriate figures in ref. 25 (see text). Such tensorial scheme (beyond being a realistic representation of neuronal networks) can actually perform, via covariant embedding procedures followed by space-time metric transformations, the fundamental operations of a two-to-three-dimensional sensorimotor network. Pathways in this scheme carry the neuronal information vectorially (covariants shown in blue, contravariants in red). The cortical circuitries perform tensor transformations. For further explanation, see text.
TENSORIAL SCHEME OF A SENSORIMOTOR NEURONAL NETWORK

Sensorimotor covariant embedding (perception - motor intention)

Contravariant sensory reception

Covariant proprioception

Sensory coordination

(reception - perception)

\[ v_{i\Omega} \cdot g^{i\omega j} = v_{j\Omega} \]

Contravariant perception

Covariant intention

\[ u_{nt'} \cdot g^{nt'mT} = u^{mT} \]

Covariant sensory reception

Contravariant motor execution
As for understanding the organization of the brain, the conceptual ‘slicing’ of CNS functioning into separable simple reflexes by Descartes in the 17th century is contrasted with the unification efforts in both morphology and physiology at the turn of this century. Ramón y Cajal attempted to bridge the gap between the complexity of entire networks and the symbolism of neuronal loops (cf. Figs 1A and 1C). At about the same time, Sherrington envisioned brain function in the form of activation patterns over large fields of neurons as an ‘enchanted loom’. However, a formal merging of structuro-functional features could not be attained at their time for lack of an adequate mathematical (geometrical) approach.

In this paper, we have attempted to relate brain organization to space-time geometry. We have assumed the unified view of space-time, as expressed by contemporary tensor formulation, which is a general treatment of abstract geometries. Indeed, it appears that the parallel, distributed structuro-functional features of neuronal networks do furnish the CNS with an innate a priori propensity to implement geometries. Such an innate functional geometry, however, is Kantian only in an extended sense, being isomorphic not with the physical space only, but with the geometry of the physical space-time continuum.

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